2. Quadratic Equations

- **Identification of quadratic equations**

Example: Check whether the following are quadratic equations or not.

(i) $(2x + 3)^2 = 12x + 3$

(ii) $x(x + 3) = (x + 1)(x - 5)$

Solution:

$$(\textbf{i})\ (2x + 3)^2 = 12x + 3$$
$$\Rightarrow 4x^2 + 12x + y = 12x + 3$$
$$\Rightarrow 4x^2 + 6 = 0$$

It is of the form $ax^2 + bx + c = 0$, where $a = 4$, $b = 0$ and $c = 6$

Therefore, the given equation is a quadratic equation

$$(\textbf{ii})\ x(x + 3) = (x + 1)(x - 5)$$
$$\Rightarrow x^2 + 3x = x^2 + x - 5x - 5$$
$$\Rightarrow 7x + 5 = 0$$

It is not of the form $ax^2 + bx + c = 0$, since the maximum power (or degree) of equation is 1.

Therefore, the given equation is not a quadratic equation.

- **Express given situation mathematically**

Example 1:

An express train takes 2 hour less than a passenger train to travel a distance of 240 km. If the average speed of the express train is 20 km/h more than that of a passenger train, then form a quadratic equation to find the average speed of the express train?

Solution:

Let the average speed of the express train be x km/h.

Since it is given that the speed of the express train is 20 km/h more than that of a passenger train,

Therefore, the speed of the passenger train will be $x - 20$ km/h.

Also we know that $\text{Time} = \dfrac{\text{Distance}}{\text{Speed}}$

Time taken by the express train to cover 240 km $= \dfrac{240}{x}$

Time taken by the passenger train to cover 240 km $= \frac{240}{x-20}$

And the express train takes 2 hour less than the passenger train. Therefore,

$$\frac{240}{x-20} - \frac{240}{x} = 2$$

$$\Rightarrow 240\left[\frac{x-(x-20)}{x(x-20)}\right] = 2$$

$$\Rightarrow 120\left(\frac{20}{x^2-20x}\right) = 1$$

$$\Rightarrow 2400 = x^2 - 20x$$

$$\Rightarrow x^2 - 20x - 2400 = 0$$

This is the required quadratic equation.

- **Solution of Quadratic Equation by Factorization Method**

  If we can factorize $ax^2+bx+c=0$, where $a \neq 0$, into a product of two linear factors, then the roots of this quadratic equation can be calculated by equating each factor to zero.

  Example:
  Find the roots of the equation, $2x^2-7\sqrt{3}x+15=0$, by factorisation.

  Solution:

  $$2x^2-7\sqrt{3}x+15=0$$

  $$\Rightarrow 2x^2-2\sqrt{3}x-5\sqrt{3}x+15=0$$

  $$\Rightarrow 2x\left(x-\sqrt{3}\right)-5\sqrt{3}\left(x-\sqrt{3}\right)=0$$

  $$\Rightarrow \left(x-\sqrt{3}\right)\left(2x-5\sqrt{3}\right)=0$$

  $$\Rightarrow \left(x-\sqrt{3}\right)=0 \text{ or } \left(2x-5\sqrt{3}\right)=0$$

  $$\Rightarrow x=\sqrt{3} \text{ or } x=\frac{5\sqrt{3}}{2}$$

  Therefore, $\sqrt{3}$ and $\frac{5\sqrt{3}}{2}$ are the roots of the given quadratic equation.

- **Solution of Quadratic Equation by completing the square**

  A quadratic equation can also be solved by the method of completing the square.

  Example:
  Find the roots of the quadratic equation, $5x^2+7x-6=0$, by the method of completing the square.

  Solution:

$$5x^2+7x-6=0$$
$$\Rightarrow 5\left[x^2+\frac{7}{5}x-\frac{6}{5}\right]=0$$
$$\Rightarrow x^2+2\times x\times\frac{7}{10}+\left(\frac{7}{10}\right)^2-\left(\frac{7}{10}\right)^2-\frac{6}{5}=0$$
$$\Rightarrow\left(x+\frac{7}{10}\right)^2-\frac{49}{100}-\frac{6}{5}=0$$
$$\Rightarrow\left(x+\frac{7}{10}\right)^2=\frac{169}{100}$$
$$\Rightarrow\left(x+\frac{7}{10}\right)=\pm\sqrt{\frac{169}{100}}=\pm\frac{13}{10}$$
$$\Rightarrow x+\frac{7}{10}=\frac{13}{10}\text{ or }x+\frac{7}{10}=-\frac{13}{10}$$
$$\Rightarrow x=\frac{13}{10}-\frac{7}{10}=\frac{3}{5}\text{ or }x=-\frac{13}{10}-\frac{7}{10}=-2$$

Therefore, –2 and $\frac{3}{5}$ are the roots of the given quadratic equation.

- **Quadratic Formula to find solution of quadratic equation:**

The roots of the quadratic equation, $ax^2+bx+c=0$, are given by, $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, where $b^2-4ac\geq 0$

Example:
Find the roots of the equation, $2x^2-3x-44=0$, if they exist, using the quadratic formula.

Solution:

$$2x^2-3x-44=0$$

Here, a = 2, b = –3, c = –4

$$\therefore b^2-4ac=(-3)^2-4\times 2\times(-44)=9+352=361>0$$

The roots of the given equation are given by $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

$$\Rightarrow x=\frac{-(-3)\pm\sqrt{361}}{2\times 2}=\frac{3\pm 19}{4}$$
$$\Rightarrow x=\frac{3+19}{4}=\frac{11}{2}\text{ or }x=\frac{3-19}{4}=-4$$

The roots are –4 and $\frac{11}{2}$.

- **Nature of roots of Quadratic Equation**

For the quadratic equation, $ax^2+bx+c=0$, where $a \neq 0$, the discriminant 'D' is defined as
$D=b^2-4ac$

The quadratic equation, $ax^2+bx+c=0$, where $a \neq 0$, has

1. two distinct real roots, if $D=b^2-4ac>0$
2. two equal real roots, if $D=b^2-4ac=0$
3. has no real roots, if $D=b^2-4ac<0$

Example: Determine the nature of the roots of the following equations

(a) $2x^2+5x-117=0$

(b) $3x^2+5x+6=0$

Solution:

(a) Here, $a = 2$, $b = 5$, $c = -117$

$\therefore D = b^2-4ac=5^2-4\times 2\times(-117)=25+936=961>0$

Therefore, the roots of the given equation are real and distinct.

(b) Here, $a = 3$, $b = 5$, $c = 6$

$\therefore D = b^2-4ac=5^2-4\times 3\times 6=25-72=-47<0$

Therefore, the roots of the given equation are not real.

• An equation which is not in the form of a quadratic equation can be reduced to a quadratic equation by proper substitution of new variables.

For example, $6x^2+\dfrac{2}{x^2}=7$

$6x^4+2=7x^2$ (On multiplying both sides by x^2)
$\Rightarrow 6x^4-7x^2+2=0$

Let $x^2=a$, we get

6a2-7a+2=0⇒6a2-4a-3a+2=0⇒2a3a-2-13a-2=0⇒3a-22a-1=0⇒a=23 or a=12

Substituting $a=x^2$, we get

$$x^2=\frac{2}{3} \text{ or } x^2=\frac{1}{2}$$
$$\Rightarrow x=\pm\sqrt{\frac{2}{3}} \text{ or } x=\pm\frac{1}{\sqrt{2}}$$

• Identities used in solving such equations are:

(1) $x^2+\frac{1}{x^2}=\left(x+\frac{1}{x}\right)^2-2$

(2) $x^2+\frac{1}{x^2}=\left(x-\frac{1}{x}\right)^2+2$

- **Relationship between zeroes and Coefficients of a polynomial**
- **Linear Polynomial**

The zero of the linear polynomial, $ax+b$, is $\frac{-b}{a}=\frac{-(\text{Constant term})}{\text{Coefficient of } x}$

Example: $3x-5$

$$3x-5=0\Rightarrow x=\frac{5}{3}$$

Zero of $3x-5$ is $\frac{5}{3}=\frac{-(-5)}{3}=\frac{-(\text{Constant term})}{\text{Coefficient of } x}$

- **Quadratic Polynomial**

If α and β are the zeroes of the quadratic polynomial, $p(x)=ax^2+bx+c$, then $(x-\alpha),(x-\beta)$ are the factors of $p(x)$.

$p(x)=ax^2+bx+c=k\left[x^2-(\alpha+\beta)x+\alpha\beta\right]$, where $k\neq 0$ is constant.

Sum of zeroes = $\alpha+\beta=\frac{-b}{a}=-\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$

Product of zeroes = $\alpha\beta=\frac{c}{a}=\frac{\text{Constant term}}{\text{Coefficient of } x^2}$

Example:

Find the zeroes of the quadratic polynomial, $2x^2+17x-9$, and verify the relationship between the zeroes and the coefficients.

Solution:

$$\begin{aligned} p(x)&=2x^2+17x-9\\ &=2x^2+18x-x-9\\ &=2x(x+9)-1(x+9)\\ &=(x+9)(2x-1)\end{aligned}$$

The zeroes of $p(x)$ are given by,

$$p(x)=0$$
$$\Rightarrow (x+9)(2x-1)=0$$
$$\Rightarrow 2x-1=0 \text{ or } x+9=0$$
$$\Rightarrow x=\frac{1}{2} \text{ or } x=-9$$

Zeroes of $p(x)$ are $\alpha=\frac{1}{2}$ and $\beta=-9$

Sum of zeroes $=\alpha+\beta=\frac{1}{2}-9=\frac{-17}{2}=-\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$

Product of zeroes $=\alpha\beta=\frac{1}{2}\times-9=\frac{-9}{2}=-\frac{\text{Constant term}}{\text{Coefficient of } x^2}$

- **Formation of Polynomial using the Sum and Product of Zeroes**

Example:

Find a quadratic polynomial, the sum and the product of whose zeroes are $\frac{-14}{3}$ and $\frac{-5}{3}$.

Solution:
Given that,
$\alpha+\beta=\frac{-14}{3}$, $\alpha\beta=\frac{-5}{3}$
The required polynomial is given by,

$$p(x)=k\left[x^2-(\alpha+\beta)x+\alpha\beta\right]$$
$$=k\left[x^2-\left(\frac{-14}{3}\right)x+\left(\frac{-5}{3}\right)\right]=k\left[x^2+\frac{14}{3}x-\frac{5}{3}\right]$$

For $k=3$,

$$p(x)=3\left[x^2+\frac{14}{3}x-\frac{5}{3}\right]=3x^2+14x-5$$

One of the quadratic polynomials, which fit the given condition, is $3x^2+14x-5$.

- **Cubic polynomial**

If α, β, γ are the zeroes of the cubic polynomial, $f(x)=ax^3+bx^2+cx+d$, then $(x-\alpha), (x-\beta), (x-\gamma)$ are the factors of $f(x)$.

$f(x)=ax^3+bx^2+cx+d=k\left[x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)x-\alpha\beta\gamma\right]$, where k is a non-zero constant

$$\alpha+\beta+\gamma=-\frac{b}{a}=-\frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3}$$

$$\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}=\frac{\text{Coefficient of } x}{\text{Coefficient of } x^3}$$

$$\alpha\beta\gamma=\frac{-d}{a}=-\frac{\text{Constant term}}{\text{Coefficient of } x^3}$$

• If α and β are the roots of $p(x) = ax^2 + bx + c$, then

Sum of roots = $\alpha + \beta = \frac{-b}{a}$
Product of roots = $\alpha\beta = \frac{c}{a}$

• If the roots of a quadratic equation $q(x)$ are known, then it can formed as follows:

$$\boxed{q(x) = x^2 - (\text{sum of roots})\, x + \text{product of roots}}$$